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Scattering of scalar waves by an obstacle: a different approach

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Abstract

We present a different approach to dealing with integral equations developed to tackle the scattering by obstacles of scalar harmonic waves. The consistency of this approach is made evident by proving that classical results are obtained for the scattering of harmonic plane and spherical waves on perfectly reflecting planes and cylinders. Then, this approach illustrated on the scattering of harmonic waves by obstacles with a surface impedance supplies a Fredholm-like equation whose solution is obtained by using the Rayleigh–Gans iterative process.

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1. Introduction

Integral equations with Green functions as kernels are used in physics to investigate the scattering by obstacles of acoustic, optical and electromagnetic waves when the boundary conditions on the total field (incident plus scattered on the obstacle S) are known [1–4]. They were developed by Weber and Helmholtz, respectively, for two-dimensional (2D) and three-dimensional (3D) problems [1, 2] and applied to the scattering of scalar harmonic fields. But while this formulation is largely based, at least for 3D problems, on the Huygens principle [5], integral equations are also a tool for solving boundary value problems of partial differential equations [6] leading to some kind of Fredholm equation. This is the way that the integral formulation is used in this work to analyse the scattering of scalar harmonic waves, solutions of the Helmholtz equation, by a plane and by a circular cylinder. The consistency of this approach is checked for harmonic plane and spherical waves incident on a perfectly reflecting smooth plane. Then, an application to scattering by impedance planes is developed, a problem which has been the object of many works in electromagnetism [7, 8] and acoustics [9] and which leads here to a Fredholm-like integral equation. The solution of this equation requires some method of successive approximations [6] among which the Rayleigh–Gans iterative process [1], also known as the Born approximation in quantum mechanics, is the most suitable. The comparison between Fredholm and conventional approaches to scattering is also discussed from a theoretical point of view, but at this stage of the investigation, only simple numerical calculations are made, further works are needed in this domain.

2. A different approach to dealing with integral equations

2.1. Plane obstacle

To avoid later confusion, we clarify some notation: $\boldsymbol{x}(xy, z)$ and $\boldsymbol{x}'(x', y', z')$ denote, respectively, the action and point sources in the Green functions, while the surface S is written as Σ, Σ', for the action and source points, respectively.

From the scalar Helmholtz equations for the field ψ and for the Green function G:

$$\Delta\psi(\boldsymbol{x}) + k^2\psi(\boldsymbol{x}) = 0 \qquad \Delta G(\boldsymbol{x}, \boldsymbol{x}') + k^2 G(\boldsymbol{x}, \boldsymbol{x}') = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{1}$$

in which Δ is the Laplacian operator and δ is the Dirac distribution, we obtain for a solution ψ inside a volume V the integral equation

$$\psi(\boldsymbol{x}) = \int_V \mathrm{d}x' \,[G(\boldsymbol{x}, \boldsymbol{x}')\Delta\psi(\boldsymbol{x}') - \psi(\boldsymbol{x}')\Delta G(\boldsymbol{x}, \boldsymbol{x}')]. \tag{2}$$

The Green theorem transforms (2) into

$$\psi(\boldsymbol{x}) = \int_S \mathrm{d}s \,[G(\boldsymbol{x}, \boldsymbol{x}')\,\partial_n\psi(\boldsymbol{x}') - \psi(\boldsymbol{x}')\partial_n G(\boldsymbol{x}, \boldsymbol{x}')] \tag{3}$$

in which ∂_n is the outward normal derivative to the surface S bounding the volume V, ψ and G satisfying some boundary condition on S. For a plane S located at $z = 0$, $-\infty < x, y < \infty$, equation (3) becomes

$$\psi(\boldsymbol{x}) = \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[G(\boldsymbol{x}, \boldsymbol{x}')\,\partial_{z'}\psi(\boldsymbol{x}') - \psi(\boldsymbol{x}')\partial_{z'}G(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0}. \tag{4}$$

When equation (4) is applied to scattering, $\psi(\boldsymbol{x})$ is considered as the total incident plus scattered field

$$\psi(\boldsymbol{x}) = \psi_i(\boldsymbol{x}) + \psi_s(\boldsymbol{x}) \tag{5}$$

and when S is a perfectly reflecting smooth plane, ψ and G satisfy on the Σ-plane, $z = 0$, the Dirichlet or the Neumann boundary conditions (soft or hard in acoustics)

$$[\psi(\boldsymbol{x})]_\Sigma = 0 \qquad [G_D(\boldsymbol{x}, \boldsymbol{x}')]_\Sigma = 0 \tag{6a}$$

$$[\partial_z\psi(\boldsymbol{x})]_\Sigma = 0 \qquad [\partial_z G_N(\boldsymbol{x}, \boldsymbol{x}')]_\Sigma = 0 \tag{6b}$$

so that we obtain from (4) the two Fredholm-like integral equations

$$\psi(\boldsymbol{x}) = \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[G_D(\boldsymbol{x}, \boldsymbol{x}')\partial_{z'}\psi(\boldsymbol{x}')]_{z'=0} \tag{7a}$$

$$\psi(\boldsymbol{x}) = -\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[\psi(\boldsymbol{x}')\partial_{z'}G_N(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0}. \tag{7b}$$

Both equations are valid for $z > 0$ (respectively, $z < 0$) corresponding to the incident field propagating in the region $z > 0$ (respectively, $z < 0$). Now, let $G(\boldsymbol{x}, \boldsymbol{x}')$ denote the free-space Green function of the Helmholtz equation: using the Weyl representation [10, 11] we obtain

$$G(\boldsymbol{x}, \boldsymbol{x}') = \left(\mathrm{i}/8\pi^2\right)\int_{-\infty}^{\infty}\mathrm{d}\gamma\int_{-\infty}^{\infty}\mathrm{d}\beta\,k_z^{-1}\exp[\mathrm{i}\beta(x - x') + \mathrm{i}\gamma(y - y') + \mathrm{i}k_z|z - z'|] \tag{8a}$$

$$k_z = \left(k^2 - \beta^2 - \gamma^2\right)^{1/2} \tag{8b}$$

and $G_{D,N}$ are obtained from G by the method of images [1–4]: let $\xi(x, y, -z)$ be the image point of $\boldsymbol{x}$ with respect to the Σ-plane $z = 0$, then

$$G_D(\boldsymbol{x}, \boldsymbol{x}') = G(\boldsymbol{x}, \boldsymbol{x}') - G(\xi, \boldsymbol{x}') \qquad G_N(\boldsymbol{x}, \boldsymbol{x}') = G(\boldsymbol{x}, \boldsymbol{x}') + G(\xi, \boldsymbol{x}') \tag{9}$$

which allows us to determine the integral equations (7*a*) and (7*b*).

The integral equation (7*a*) (respectively, (7*b*)) concerns pseudoscalar (respectively, scalar) waves: in the former case, relation (5) becomes $\psi = \psi_i - \psi_r$ and in the latter $\psi = \psi_i + \psi_r$, in which ψ_r denotes the reflected wave.

Let us now make clear the differences between the present approach and the conventional one in which to avoid confusion we denote the Green functions by g. The total field ψ still satisfies the boundary conditions (6*a*) and (6*b*), while the corresponding boundary conditions for $g_{D,N}$ are now given on the Σ'-plane $z' = 0$ and no longer on Σ,

$$[g_D(\boldsymbol{x}, \boldsymbol{x}')]_{\Sigma'} = 0 \qquad [\partial_z g_N(\boldsymbol{x}, \boldsymbol{x}')]_{\Sigma'} = 0 \tag{10}$$

supplying from (4) the two integral equations

$$\psi(\boldsymbol{x}) = \psi_i(\boldsymbol{x}) + \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[g_N(\boldsymbol{x}, \boldsymbol{x}')\,\partial_{z'}\psi_s(\boldsymbol{x}')]_{z'=0} \tag{11a}$$

$$\psi(\boldsymbol{x}) = \psi_i(\boldsymbol{x}) - \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[\psi_s(\boldsymbol{x}')\partial_{z'} g_D(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0} \tag{11b}$$

in which $g_{D,N}$ are obtained from the free-space Green function g, still by the method of images but now taken with respect to the Σ'-plane $z' = 0$ so that with $\xi'(x', y'; -z')$ we obtain

$$g_D(\boldsymbol{x}, \boldsymbol{x}') = g(\boldsymbol{x}, \boldsymbol{x}') - g(\boldsymbol{x}, \xi') \qquad g_D(\boldsymbol{x}, \boldsymbol{x}') = g(\boldsymbol{x}, \boldsymbol{x}') + g(\boldsymbol{x}, \xi') \tag{12}$$

where g is the spherical wave (a heritage of the Huygens principle)

$$g(\boldsymbol{x}, \boldsymbol{x}') = (1/4\pi)\exp(\mathrm{i}k|\boldsymbol{x} - \boldsymbol{x}'|)/|\boldsymbol{x} - \boldsymbol{x}'|. \tag{12a}$$

If $\psi_i(\boldsymbol{x})$ or $\partial_z\psi_i(\boldsymbol{x})$ is known on the plane $z = 0$, then using the boundary conditions (6*a*) and (6*b*), the expressions of $\psi_s(\boldsymbol{x}')$ to be introduced in the integrand of (11*a*) and (11*b*) are also known (an excellent discussion is given in [2]). So, strictly speaking, equations (11*a*) and (11*b*) are not integral equations, but they are solutions of the Helmholtz equation in an integral form. Otherwise, to obtain $\psi_s(\boldsymbol{x}')$, one has to deal with an inhomogeneous Fredholm integral equation of the second kind as discussed in section 3.1.2.

2.2. *Cylindrical obstacle*

When the obstacle S is a circular cylinder of radius a, the boundary conditions (6*a*) and (6*b*) become with $\boldsymbol{r} = (r, \phi)$ (we leave aside the case of a parabolic cylinder which could be analysed in the same way)

$$\psi(\boldsymbol{r})|_{r=a} = 0 \qquad G(\boldsymbol{r}, \boldsymbol{r}')|_{r=a} = 0 \tag{13a}$$

$$\partial_r\psi(\boldsymbol{r})|_{r=a} = 0 \qquad \partial_r G(\boldsymbol{r}, \boldsymbol{r}')|_{r=a} = 0 \tag{13b}$$

and the integral equations outside the cylinder are

$$\begin{aligned} \psi(\boldsymbol{r}) &= \int_0^{2\pi} a\,\mathrm{d}\phi'\,[G_D(\boldsymbol{r}, \boldsymbol{r}')\,\partial_{r'}\psi(\boldsymbol{r}')]_{r'=a} \qquad r \geqslant a \\ \psi(\boldsymbol{r}) &= -\int_0^{2\pi} a\,\mathrm{d}\phi'\,[\psi(\boldsymbol{r}')\,\partial_{r'} G_N(\boldsymbol{r}, \boldsymbol{r}')]_{r'=a} \qquad r \geqslant a. \end{aligned} \tag{14}$$

The free-space Green function $G(\boldsymbol{r}, \boldsymbol{r}')$ is the expansion of the Hankel function $\mathrm{i}/4\pi H_0^{(1)}(k|\boldsymbol{r} - \boldsymbol{r}'|)$ [3] in terms of Bessel and Hankel functions J_m, H_m (written for $H_m^{(1)}$) with $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \geqslant 1$,

$$4\mathrm{i}\pi G(\boldsymbol{r}, \boldsymbol{r}') = \sum_{m=0}^{\infty} \varepsilon_m \cos[m(\phi - \phi')]\,J_m(kr')\,H_m(kr) \qquad r > r'. \tag{15}$$

From now on, we write $\sum$ for $\sum_{m=0}^{\infty}$ (not to be confused with the plane Σ) and

$$C_m = \varepsilon_m \cos[m(\phi - \phi')]. \tag{15a}$$

Then, we look for G_D in the form [6]

$$G_D(\boldsymbol{r}, \boldsymbol{r}') = G(\boldsymbol{r}, \boldsymbol{r}') + \gamma(\boldsymbol{r}, \boldsymbol{r}') \tag{16}$$

in which $\gamma(\boldsymbol{r}, \boldsymbol{r}')$ is a solution of the 2D Helmholtz equation $(\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\phi^2 + k^2)\gamma(\boldsymbol{r}, \boldsymbol{r}') = 0$ such that the expression (16) satisfies the boundary condition (13*a*). The convenient solutions are

$$\gamma(\boldsymbol{r}, \boldsymbol{r}') = \sum a_m C_m J_m(kr')\, J_m(kr). \tag{17}$$

Taking into account (17) and substituting (16) into (13*a*) gives the amplitudes a_m and we obtain

$$G_D(\boldsymbol{r}, \boldsymbol{r}') = \mathrm{i}/4\pi \sum C_m J_m(kr')[H_m(kr) - J_m(kr)H_m(ka)/J_m(ka)] \qquad r \geqslant r' \tag{18a}$$

and similarly

$$G_N(\boldsymbol{r}, \boldsymbol{r}') = \mathrm{i}/4\pi \sum C_m J_m(kr')[H_m(kr) - J_m(kr)H'_m(ka)/J'_m(ka)] \qquad r \geqslant r' \tag{18b}$$

in which H' and J' are the derivatives of the Hankel and Bessel functions.

2.3. *Consistency of Fredholm integral equations*

To check the consistency of this approach, we consider a scalar harmonic plane wave (the time dependence $\exp(\mathrm{i}\omega t)$ is implicit) impinging from the region $z < 0$ on the $z = 0$ plane,

$$\psi_i(\boldsymbol{x}) = \exp[\mathrm{i}k(x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta)] \tag{19}$$

so that, according to the Descartes–Snell law, the reflected field is

$$\psi_r(\boldsymbol{x}) = \exp[\mathrm{i}k(x \sin\theta \cos\phi + y \sin\theta \sin\phi - z \cos\theta)]. \tag{19a}$$

Since ψ_i is a scalar the total field ψ is $\psi_i + \psi_r$ and one has to check that ψ is a solution of equation (7*b*).

Now, according to (19) and (19*a*), we obtain

$$[\psi(\boldsymbol{x}')]_{z'=0} = 2\exp[\mathrm{i}k(x' \sin\theta \cos\phi + y' \sin\theta \sin\phi] \tag{20}$$

while from (8) and (9)

$$G_N(\boldsymbol{x}, \boldsymbol{x}') = (\mathrm{i}/8\pi^2) \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma\, k_z^{-1} \exp[\mathrm{i}\beta(x - x') + \mathrm{i}\gamma(y - y')]$$
$$\times \{\exp[\mathrm{i}k_z|z - z'|] + \exp[\mathrm{i}k_z|z + z'|]\} \tag{21}$$

and a simple calculation gives

$$\partial_{z'} G_N(\boldsymbol{x}, \boldsymbol{x}') = -(1/8\pi^2) \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma \exp[\mathrm{i}\beta(x - x') + \mathrm{i}\gamma(y - y')]$$
$$\times \{\exp[\mathrm{i}k_z|z - z'|]\partial_{z'}|z - z'| + \exp[\mathrm{i}k_z|z + z'|]\partial_{z'}|z + z'|\} \tag{21a}$$

and using the relations (z is negative for the action point and positive for its image)

$$\begin{array}{lll} |z - z'|_{z'=0} = -z & [\partial_{z'}|z - z'|]_{z'=0} = 1 & z < 0 \\ |z + z'|_{z'=0} = z & [\partial_{z'}|z + z'|]_{z'=0} = 1 & z > 0 \end{array} \tag{22}$$

we obtain

$$[\partial_{z'} G_N(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0} = -\left(1/4\pi^2\right) \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma \exp[\mathrm{i}\beta(x - x') + \mathrm{i}\gamma(y - y')] \cos(k_z z). \tag{23}$$

Substituting (20) and (23) into the right-hand side of (7b) gives

$$\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[\psi(\boldsymbol{x}')\partial_{z'}G_N(\boldsymbol{x},\boldsymbol{x}')]_{z'=0}$$
$$= \left(-1/2\pi^2\right)\iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\,\exp(\mathrm{i}\beta x+\mathrm{i}\gamma y)\cos(k_z)F(\beta,\gamma) \qquad (24)$$

$$F(\beta,\gamma) = \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,\exp[-\mathrm{i}x'(\beta+k\sin\theta\cos\phi)-\mathrm{i}y'(\gamma+k\sin\theta\sin\phi)]$$
$$= 4\pi^2\delta(\beta+k\sin\theta\cos\phi)\delta(\gamma+k\sin\theta\sin\phi) \qquad (24a)$$

where δ is the Dirac distribution. Substituting (24a) into (23) and using the definition (8b) of k_z we obtain

$$\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[\psi(\boldsymbol{x}')\partial_{z'}G_N(\boldsymbol{x},\boldsymbol{x}')]_{z'=0} = -2\iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\,\exp(\mathrm{i}\beta x+\mathrm{i}\gamma y)$$
$$\times\cos(kz)\delta(\beta+k\sin\theta\cos\phi)\delta(\gamma+k\sin\theta\sin\phi)$$
$$= -2\exp[\mathrm{i}k(x\sin\theta\cos\phi+y\sin\theta\sin\phi]\cos(kz) \qquad (25)$$

which is $-\psi(\boldsymbol{x})$ according to the left-hand side of (7b) and proves the correctness of the Fredholm integral equation. The consistency of this approach for spherical waves incident on a plane mirror is shown in appendix A, and the case of plane waves impinging on a circular cylinder is also easily tackled.

Now that the consistency of this integral equation approach is checked we may investigate scattering problems on a plane with a surface impedance which has been the object, as mentioned in the introduction, of many works in electromagnetism [7, 8] and in acoustics [9].

2.4. Numerical application

A numerical evaluation of the 2D integral equation

$$\psi(x,z) = (1/2\pi)\int_{-\infty}^{\infty} \mathrm{d}\beta\,\exp(\mathrm{i}\beta x)\cos(k_z z)F(\beta)$$
$$F(\beta) = \int_{-\infty}^{\infty} \mathrm{d}x'\,\exp(-\mathrm{i}\beta x')\psi(x',0)$$

is made when the total field on the Σ'-plane is the Gaussian function $\exp(-x'^2/d^2)$, corresponding to an incident Gaussian beam and $k_z = (d^2-\beta^2)^{1/2}$. See tables 1 and 2. The calculations present no particular difficulties.

Table 1. x and z data for $d = 1$.

	x (m)		
z (km)	$\frac{1}{2}$	1	2
0.1	0.690	0.321	−0.2268
0.5	0.689	0.190	−0.2260
1	0.642	−0.434	−0.1760

Table 2. x and z data for $d = 2$.

	x (m)		
z (km)	0	1	2
0.1	0.870	-5.49×10^{-3}	-0.120
0.5	0.527	-5.59×10^{-3}	-0.090
1	0.023	-5.38×10^{-3}	-8.95×10^{-3}

3. Scattering by impedance planes

3.1. TM electromagnetic plane wave

As is well known [14, 15], when the electromagnetic field does not depend on one coordinate, say y, the Maxwell equations divide into two independent sets supplying the TM and TE components which can be obtained in terms of a scalar solution of the 2D Helmholtz equation. Introducing the variable $\boldsymbol{u}(x, z)$ the boundary conditions (6a) and (6b), and the integral equations (7a) and (7b) are still valid with $\psi(\boldsymbol{u})$ and $G_{D,N}(\boldsymbol{u}, \boldsymbol{u}')$ when S is a perfectly conducting smooth plane, one has just to set $y = 0$ and $\gamma = 0$ in these relations.

Suppose now that a TM harmonic plane wave (still with $\exp(\mathrm{i}\omega t)$ implicit)

$$\psi_i(\boldsymbol{u}) = \exp[-\mathrm{i}\omega/c(x\sin\theta + z\cos\theta)] \tag{26}$$

impinges from the region $z < 0$ on a surface impedance sheet located at $z = 0$ and obtained by coating a perfectly conducting plane with a thin layer of a dielectric with thickness d and permittivity ε. The face $z = 0_+$ is covered with dielectric, the face $z = 0_-$ is in free space and the face $z = d$ is perfectly conducting (figure 1).

3.1.1. Fredholm integral equation. To get the Fredholm integral equation satisfied by the total field $\psi(\boldsymbol{u})$ one has first to obtain the boundary conditions on the face $z = 0$. Inside the dielectric, Maxwell's equations for a TM harmonic field

$$\partial_z H_y = -\mathrm{i}c^{-1}\omega\varepsilon E_x \qquad \partial_x H_y = \mathrm{i}c^{-1}\omega\varepsilon E_z \qquad \mathrm{i}c^{-1}\omega\partial_t H_y = \partial_x E_z - \partial_z E_x \tag{27}$$

have the following solution satisfying the Neumann boundary condition $\partial_z H_y = 0$ on the face $z = d$ of the layer:

$$\begin{aligned} H_y &= \exp[\mathrm{i}(\omega t - \chi_x x)\cos[\chi_z(z-d)] \\ E_x &= -(\mathrm{i}c/\omega\varepsilon)\chi_z \exp[\mathrm{i}(\omega t - \chi_x x)]\sin[\chi_z(z-d)] \end{aligned} \tag{28}$$

with

$$\chi_x^2 + \chi_z^2 = \varepsilon k^2 \qquad k = \omega/c. \tag{28a}$$

Now, in free space ($\varepsilon = 1$), we may write the components $\mathcal{H}_y$, $\mathcal{E}_x$, of the TM field

$$\begin{aligned} \mathcal{H}_y &= \exp[\mathrm{i}(\omega t - k_x x)][A\exp(\mathrm{i}k_z z) + B\exp(-\mathrm{i}k_z z)] \\ \mathcal{E}_x &= (c/\omega)k_z \exp[\mathrm{i}(\omega t - k_x x)][A\exp(\mathrm{i}k_z z) - B\exp(-\mathrm{i}k_z z)] \end{aligned} \tag{29}$$

with

$$k_x^2 + k_z^2 = k^2 \tag{29a}$$

in which the amplitudes A and B, are obtained from the continuity conditions

$$(H_y)_{z=0_+} = (\mathcal{H}_y)_{z=0_-} \qquad (E_x)_{z=0_+} = (\mathcal{E}_x)_{z=0_-}. \tag{30}$$

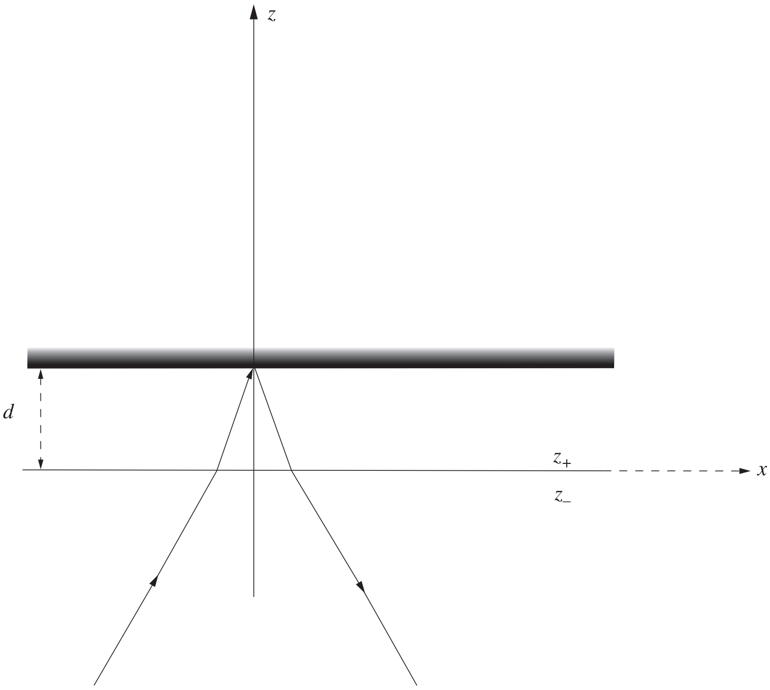

Figure 1. Reflection at an impedance plane.

Substituting (28) and (29) into (30), we first obtain the Descartes–Snell condition $k_x = \chi_x$ and then the two relations

$$A + B = \cos(\chi_z d) \qquad A - B = \mathrm{i}(\chi_z/\varepsilon k_z)\sin(\chi_z d). \tag{31}$$

From (29) and (30), we obtain the boundary condition on the face $z = 0_-$,

$$\left[\cos(\chi_z d)\partial_z \mathcal{H}_y + c^{-1}\chi_z \sin(\chi_z d)\mathcal{H}_y\right]_{z=0_-} = 0 \tag{32}$$

which is a particular case of a general result obtained by Idemen [16] for a plane coated with multi-layer sheets. Assuming $\chi_z d \ll 1$, the expression (32) becomes

$$\left[\partial_z \mathcal{H}_y + d\varepsilon^{-1}\chi_z^2 \mathcal{H}_y\right]_{z=0_-} = 0 \tag{33}$$

with according to (28a) and (29a) since $k_x^2 = \chi_x^2$

$$\chi_z^2 = k_z^2 + (\varepsilon - 1)k^2. \tag{33a}$$

Then, denoting by $\psi(\boldsymbol{u})$ the component $\mathcal{H}_y$ of the TM field we may write the boundary condition (33) as

$$[\partial_z \psi(\boldsymbol{u}) + N\psi(\boldsymbol{u})]_{z=0_-} = 0 \qquad \varepsilon N = d\left[k_z^2 + (\varepsilon - 1)k^2\right] \tag{34}$$

and we denote by $\mathcal{G}(\boldsymbol{u}, \boldsymbol{u}')$ the Green function satisfying the same boundary condition

$$[\partial_z \mathcal{G}(\boldsymbol{u}, \boldsymbol{u}') + N\mathcal{G}(\boldsymbol{u}, \boldsymbol{u}')]_{z=0_-} = 0. \tag{34a}$$

So, according to (7b) the Fredholm integral equation satisfied by the total field is

$$\psi(\boldsymbol{u}) = -\int_{-\infty}^{\infty} \mathrm{d}x' \, [\psi(\boldsymbol{u}')\partial_{z'}\mathcal{G}(\boldsymbol{u}, \boldsymbol{u}')]_{z'=0} \qquad z \leqslant 0 \tag{35}$$

and we have now to solve (35) which requires some method of approximation [6].

3.1.2. Solution of the Fredholm integral equation. Setting $\gamma = 0$ in (21) gives

$$G_N(\boldsymbol{u}, \boldsymbol{u}') = (\mathrm{i}/\pi)\int_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\, k_z^{-1} \exp[\mathrm{i}\beta(x-x')]\{\exp[\mathrm{i}k_z|z-z'|] + \exp[\mathrm{i}k_z|z+z'|]\} \quad (36)$$

such that $[\partial_z G_N)]_{z=0} = 0$ so that

$$\mathcal{G}(\boldsymbol{u}, \boldsymbol{u}') = G_N(\boldsymbol{u}, \boldsymbol{u}')\exp(-N|z-z'|). \quad (37)$$

To prove this result, one observes that since according to (22) $[\partial_z|z-z'|]_{z=0} = 1$ for $z' < 0$ one has

$$[\partial_z \mathcal{G}]_{z=0} = -N[\mathcal{G}]_{z=0}[\partial_z|z-z'|]_{z=0} \quad (38)$$

which is the boundary condition (34*a*) and concludes the proof.

Now, $\psi(\boldsymbol{u})$ is the total field $\psi_i(\boldsymbol{u}) + \psi_r(\boldsymbol{u})$, and introducing a new set of functions $\phi_{i,r}(\boldsymbol{u})$ we may write the integral equation (35) with all the known terms on the right-hand side

$$\psi_r(\boldsymbol{u}) + \phi_r(\boldsymbol{u}) = -[\psi_i(\boldsymbol{u}) + \phi_i(\boldsymbol{u})] \qquad z \leqslant 0 \quad (39)$$

$$\phi_{i,r}(\boldsymbol{u}) = \int_{-\infty}^{\infty} \mathrm{d}x'\,[\psi_{i,r}(\boldsymbol{u}')\partial_{z'}\mathcal{G}(\boldsymbol{u}, \boldsymbol{u}')]_{z'=0} \qquad z \leqslant 0 \quad (39a)$$

and we obtain in appendix B the following expression for $\phi_i(\boldsymbol{u})$:

$$\phi_i(\boldsymbol{u}) = -\exp(\mathrm{i}kx\sin\theta)\cos(kz\cos\theta)(1+\mathrm{i}b)\exp(\mathrm{i}kbz\sin\theta) \qquad z \leqslant 0 \quad (40)$$

$$b = kd(\varepsilon\sin\theta)^{-1}\left(\varepsilon - \sin^2\theta\right) \quad (40a)$$

that we write in terms of the function $h(b, z)$

$$\phi_i(\boldsymbol{u}) = -2^{-1}h(b,z)\left[\psi_i(\boldsymbol{u}) + \psi_r^{(0)}(\boldsymbol{u})\right] \qquad h(b,z) = (1+\mathrm{i}b)\exp(kbz\sin\theta) \quad (41)$$

in which $\psi_r^{(0)}$ is the field reflected according to the Descartes–Snell law,

$$\psi_r^{(0)}(\boldsymbol{u}) = \exp[-\mathrm{i}\omega/c(x\sin\theta - z\cos\theta)]. \quad (41a)$$

As previously mentioned, we need some method of successive approximation [6] to solve the Fredholm equation and the Rayleigh–Gans iterative process [1] is particularly suitable. So we obtain from (39)

$$\psi_r^{(n)}(\boldsymbol{u}) = -\left[\psi_i(\boldsymbol{u}) + \phi_i(\boldsymbol{u}) + \phi_r^{(n-1)}(\boldsymbol{u})\right] \qquad z \leqslant 0 \quad (42)$$

implying that the nth term in the iterative series is obtained by using the $(n-1)$th term in the expression (39*a*) of ϕ_r, while $\psi_r^{(0)}(\boldsymbol{u})$ is taken as an initial function to start the process and since in this case $b = 0$ we obtain from (39*a*) and (40) (equation (39) reduces to a simple identity for $b = 0$)

$$\phi_i^{(0)}(\boldsymbol{u}) = \phi_r^{(0)}(\boldsymbol{u}) = 2^{-1}\left[\psi_i(\boldsymbol{u}) + \psi_r^{(0)}(\boldsymbol{u})\right]. \quad (43)$$

So, for the first iteration, one obtains at once

$$\psi_r^{(1)}(\boldsymbol{u}) = -\left[\psi_i(\boldsymbol{u}) + \phi_i(\boldsymbol{u}) + \phi_r^{(0)}(\boldsymbol{u})\right] \qquad z \leqslant 0. \quad (44)$$

Substituting (43) and (41) into (42) gives with h written for $h(b, z)$,

$$\psi_r^{(1)}(\boldsymbol{u}) = 2^{-1}(1+h)\psi_r^{(0)}(\boldsymbol{u}) - 2^{-1}(1-h)\psi_i(\boldsymbol{u}) \qquad z \leqslant 0. \quad (44a)$$

The second term is also easily obtained, since from (44*a*) we obtain

$$\phi_r^{(1)}(\boldsymbol{u}) = 2^{-1}(1+h)\phi_r^{(0)}(\boldsymbol{u}) - 2^{-1}(1-h)\phi_i(\boldsymbol{u}). \quad (45)$$

Substituting (45) into (42) gives the following expression:

$$\psi_r^{(2)} = -\left[\psi_i + 2^{-1}(1+h)\left(\phi_i + \phi_r^{(0)}\right)\right] \quad (46)$$

Table 3. $b = 10^{-2}$.

	Re ψ			Im y		
n	$\pi/4$	$\pi/3$	π	$\pi/4$	$\pi/3$	π
49	0.6657	0.4707	−0.9514	1.050	1.087	−0.4854
50	0.6640	0.4695	−0.9514	1.057	1.135	−0.4951

Table 4. $b = 10^{-1}$.

	Re ψ			Im y		
n	$\pi/4$	$\pi/3$	π	$\pi/4$	$\pi/3$	π
49	−1.8631	−1.3174	2.6348	1.6682	1.5457	−1.3593
50	−1.9112	−1.3514	2.7028	1.6105	1.5048	−1.2775

Table 5. $b = 5 \times 10^{-1}$.

	Re ψ			Im y		
n	$\pi/4$	$\pi/3$	π	$\pi/4$	$\pi/3$	π
49	4.5764	3.2360	−6.4721	−2.6232	−1.4888	4.7098
50	5.4090	3.8247	−7.6495	−1.3023	−0.5548	2.8418

and still using (43) and (41), this expression becomes

$$\psi_r^{(2)} = 2^{-2}(1+h)^2\psi_r^{(0)} - \left[1 - 2^{-2}(1+h)^2\right]\psi_i \tag{46a}$$

so that the second iteration for ϕ_r is

$$\phi_r^{(2)} = 2^{-2}(1+h)^2\phi_r^{(0)} - \left[1 - 2^{-2}(1+h)^2\right]\phi_i \tag{47}$$

which supplies $\psi_r^{(3)}$ and so on. The method for obtaining the nth term is clear and we obtain

$$\psi_r^{(n)} = 2^{-n}(1+h)^n\psi_r^{(0)} - \left[1 - 2^{-n}(1+h)^n\right]\psi_i \tag{48}$$

but the convergence of this iterative process is still an open question. We remind the reader that the expression (38) of $\phi_i(\boldsymbol{u})$ is valid for $\chi_z d \ll 1$, that is, in fact, for $kd \ll 1$. The convergence of (48) is numerically checked for $\psi(z) = \exp(\mathrm{i}z)$ at $z = \pi/4, \pi/3, \pi$ and $b = 10^2, 10^{-1}, 5 \times 10^{-1}$, to $n = 50$. Tables 3–5 give Re ψ and Im ψ for $n = 49$ and 50.

So, the convergence for this numerical example requires b to be small, decreases when z approaches π and is better for the real part than for the imaginary part of ψ.

Remark. In the conventional method, the integral equation (11b) gives

$$\psi(\boldsymbol{u}) = \psi_i(\boldsymbol{u}) - \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[\psi_s(\boldsymbol{u}')\partial_{z'}\boldsymbol{g}_D(\boldsymbol{u},\boldsymbol{u}')]_{z'=0}$$

with $\boldsymbol{g}_D(\boldsymbol{u},\boldsymbol{u}') = g_D(\boldsymbol{u},\boldsymbol{u}')\exp[-N(z-z')]$ and $g_D(\boldsymbol{u},\boldsymbol{u}')$ given by (12) and (12a). When ψ_i is not known on the impedance plane, the usual approach is to express the total field in the medium of incidence in terms of an integral along the plane whose integrand contains the value of the field on the plane. Then letting the point of observation tend to the plane yields an inhomogeneous Fredholm integral equation of the second kind whose solution supplies $\psi_s(\boldsymbol{u}')$ on the impedance plane.

3.2. Acoustical spherical wave

3.2.1. Fredholm equation. We assume that an acoustical spherical wave impinges on a plane characterized by an impedance $Z(\omega)$ that depends only on frequency [9]. Introducing a parameter a with the dimension of $(\text{length})^{-1}$, we may write the boundary conditions as

$$[\{\partial_z + \mathrm{i}aZ(\omega)\}\psi(\boldsymbol{x})]_{z=0} = 0 \qquad [\{\partial_z + \mathrm{i}aG_Z(\boldsymbol{x}, \boldsymbol{x}')\}]_{z=0} = 0 \tag{49}$$

and from now on, we write Z for $Z(\omega)$. The integral equation (7b) is still valid provided that we change G_N into G_Z and we now prove (by introducing the function $E(z, z')$) that G_Z is

$$G_Z(\boldsymbol{x}, \boldsymbol{x}') = G_N(\boldsymbol{x}, \boldsymbol{x}')E(z, z') \tag{50}$$

$$E(z, z') = \exp(\mathrm{i}a|z - z'|])U(z' - z) + \exp(-\mathrm{i}a|z - z'|)U(z - z') \tag{50a}$$

in which U is the unit step function and we find at once for $z = 0$

$$[E(z, z')]_{z=0} = \exp(\mathrm{i}aZz') \tag{51}$$

and with the relation already used in (22)

$$\partial_z|z - z'| = \begin{cases} 1 & z - z' > 0 \\ -1 & z - z' < 0 \end{cases} \tag{52}$$

a simple calculation gives, since the Dirac distributions supplied by the derivatives of the unit step functions cancel out,

$$\partial_z E(z, z') = -\mathrm{i}aZE(z, z') \qquad [\partial_z E(z, z')]_{z=0} = -\mathrm{i}aZ\exp(\mathrm{i}aZz'). \tag{53}$$

Then, taking into account (51), (53) and the relation $[\partial_z G_N(\boldsymbol{x}, \boldsymbol{x}')]_{z=0} = 0$, the derivative of (50) on $z = 0$ is

$$\begin{aligned}[\partial_z G_Z(\boldsymbol{x}, \boldsymbol{x}')]_{z=0} &= [G_N(\boldsymbol{x}, \boldsymbol{x}')]_{z=0}[\partial_z E(z, z')]_{z=0} \\ &= -\mathrm{i}aZ[G_Z(\boldsymbol{x}, \boldsymbol{x}')]_{z=0}\end{aligned} \tag{54}$$

which is the boundary condition (49) and justifies the expression (50) of $G_Z(\boldsymbol{x}, \boldsymbol{x}')$.

Now we need $[\partial_{z'} G_Z(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0}$ that intervenes as the kernel in the integral equation, one easily obtains

$$[E(z, z')]_{z'=0} = \exp(-\mathrm{i}aZz) \qquad [\partial_{z'} E(z, z')]_{z'=0} = \mathrm{i}aZ\exp(-\mathrm{i}aZz) \tag{55}$$

and a simple calculation gives

$$[\partial_{z'} G(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0} = \exp(-\mathrm{i}aZz)[\partial_{z'} G_N(\boldsymbol{x}, \boldsymbol{x}') + \mathrm{i}aZG_N(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0} \tag{56}$$

so that the integral equation (7b) becomes

$$\psi(\boldsymbol{x}) = -\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y' \exp(-\mathrm{i}aZz)[\psi(\boldsymbol{x}')\{\partial_{z'} G_N(\boldsymbol{x}, \boldsymbol{x}') + \mathrm{i}aZG_N(\boldsymbol{x}, \boldsymbol{x}')]_{z'=0}. \tag{57}$$

Substituting (8a) and (A3) into (57) gives finally

$$4\pi^2 \exp(\mathrm{i}aZz)\psi(\boldsymbol{x}) = \iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\ (1 + aZ/k_z)\cos(k_z z)\exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\ H(\beta, \gamma) \tag{58}$$

$$H(\beta, \gamma) = \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y' \exp(-\mathrm{i}\beta x' - \mathrm{i}\gamma y')[\psi(x')]_{z'=0} \tag{58a}$$

in which according to (5) and (A1) in appendix A

$$\psi(\boldsymbol{x}) = |\boldsymbol{x}|\exp(\mathrm{i}k|\boldsymbol{x}|) + \psi_r(\boldsymbol{x}). \tag{59}$$

So, introducing the functions $\phi\{\psi_{i,r}\}$, we may write the integral equation in the form

$$\psi_r(\boldsymbol{x}) - \phi\{\psi_r\} = -[\psi_i(\boldsymbol{x}) - \phi\{\psi_i\}] \tag{60}$$

$$\phi\{\psi_{i,r}\} = (4\pi^2)^{-1} \exp(-\mathrm{i}aZz)[A\{\psi_{i,r}\} + aZC\{\psi_{i,r}\}] \tag{60a}$$

in which the functions A and C are

$$A\{\psi_{i,r}\} = \iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\,\cos(k_z z)\exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\, H_{\beta,\gamma}\{\psi_{i,r}\} \tag{61a}$$

$$C\{\psi_{i,r}\} = \iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\, k_z^{-1}\cos(k_z z)\exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\, H_{\beta,\gamma}\{\psi_{i,r}\} \tag{61b}$$

while the integrals $H_{\beta,\gamma}\{\psi_{i,r}\}$ are given by (58a)

$$H_{\beta,\gamma}\{\psi_{i,r}\} = \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y' \exp(-\mathrm{i}\beta x' - \mathrm{i}\gamma y')[\psi_{i,r}(\boldsymbol{x}')]_{z'=0}. \tag{62}$$

The functions A and C are made explicit in appendix C when ψ_i is the spherical wave (A1) and we now have to look for the solution of the integral equation (60).

3.2.2. Solution of the Fredholm equation. We now look for the solution $\psi_r(\boldsymbol{x})$ of the integral equation (60) in the form

$$\psi_r(\boldsymbol{x}) = \psi_r^{(0)}(\boldsymbol{x}) + \chi(\boldsymbol{x}) \tag{63}$$

so that with the definition (60a) of the function ϕ we find

$$\phi\{\psi_r\} = \phi\big\{\psi_r^{(0)}\big\} + \phi\{\chi\}. \tag{63a}$$

But from (A14) in appendix A, (C1) and (C3) in appendix C, we obtain

$$A\big\{\psi_r^{(0)}\big\} = A\{\psi_i\} \qquad C\big\{\psi_r^{(0)}\big\} = C\{\psi_i\} \tag{64}$$

which implies the following simple relation:

$$\phi\big\{\psi_r^{(0)}\big\} = \phi\{\psi_i\} \tag{64a}$$

so (63a) becomes

$$\phi\{\psi_r\} = \phi\{\psi_i\} + \phi\{\chi\}. \tag{65}$$

Substituting (63), (65) and the definition (C1) of $\psi^{(0)}$ in appendix C, into (60) gives

$$\chi(\boldsymbol{x}) - \phi\{\chi\} = -\psi^{(0)}(\boldsymbol{x}) + 2\phi\{\psi_i\} \tag{66}$$

that we write using the relation (C4) of appendix C

$$\chi(\boldsymbol{x}) - \phi\{\chi\} = \Phi\big\{\psi^{(0)}\big\} \tag{67}$$

$$\Phi\big\{\psi^{(0)}\big\} = \big(\alpha - 1 + \alpha aZ\partial_{z^\circ}^{-1}\big)\psi^{(0)}(\boldsymbol{x}). \tag{67a}$$

The solution of (67) can be obtained iteratively, still using the Rayleigh–Gans approximation, with at the nth step

$$\chi^{(n)}(\boldsymbol{x}) = \phi\big\{\chi^{(n-1)}\big\} + \Phi\big\{\psi^{(0)}\big\}. \tag{68}$$

It is shown in appendix D that the first iterated term for the impedance contribution to the solution of equation (60) is with $\Phi\big\{\psi^{(0)}\big\}$ given by (67a),

$$\chi^{(1)}(\boldsymbol{x}) = \Phi\big\{\psi^{(0)}\big\} + \alpha(\alpha-1)\big(1 + \alpha^2 aZ\partial_{z^\circ}^{-1}\big)\psi^{(0)}(x) - \mathrm{i}\alpha aZ\big(1 + \alpha aZ\partial_{z^\circ}^{-1}\big)\partial_{z^\circ}^{-1}\psi^{(0)}(\boldsymbol{x}) \tag{69}$$

an expression that depends on $aZ\partial_{z^\circ}^{-1}\psi^{(0)}$ and $a^2Z^2\partial_{z^\circ}^{-2}\psi^{(0)}$. The calculation of the following terms becomes rather tedious. But, as noted in [9], it is known [17] that the general reflection of the sound field above an impedance plane due to a point source is not amenable to easy computation. The difficulty is due to having to convert spherical waves into plane harmonic waves for which impedance is defined. There only exist some asymptotically expanded formulae [18, 19] in terms of a distance parameter formed by the horizontal distance, wavenumber and impedance so that the Fredholm integral equation could improve this situation.

4. Discussion

The Fredholm integral equation developed in this work to analyse scattering, in particular on impedance planes, has many other applications, for instance in the case of cylindrical obstacles [20] as well as for scattering by punctured planes leading to a new theory of diffraction by plane apertures [21]. Let us insist on a problem close to that discussed here: the scattering of a harmonic wave on a perfectly reflecting rough plane with a small roughness function $f(x, y)$ on the plane $z = 0$. Then, changing $z' = 0$ into $z' = f(x', y')$ the integral equation (7b) becomes

$$\psi(\boldsymbol{x}) = -\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[\psi(\boldsymbol{x}')\partial_{z'}G_N(\boldsymbol{x}, \boldsymbol{x}')]_{z'=f(x',y')}. \tag{70}$$

Neglecting the terms $|f(x', y')|^n$ for $n \geqslant 2$, the Taylor series expansion of the integrand gives

$$[\psi(\boldsymbol{x}')\partial_{z'}G_N(\boldsymbol{x}, \boldsymbol{x}')]_{z'=f(x',y')} = \chi_0(x', y') + f(x', y')\chi_1(x', y') \tag{70a}$$

where $\chi_0(x', y')$ is the integrand of (7b) so that substituting (70a) into (70) gives

$$\psi(\boldsymbol{x}) = \psi_0(\boldsymbol{x}) - \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,\chi_1(x', y')\,f(x', y') \tag{71}$$

in which $\psi_0(\boldsymbol{x})$ is the total field for reflection on a perfectly reflecting smooth plane, while $\chi_1(x', y')$ depends on the unknown field $[\psi(\boldsymbol{x}')]_{z'=0}$. The Rayleigh–Gans method, still suitable to obtain an approximation of (71), gives at the nth iteration

$$\psi^{(n)}(\boldsymbol{x}) = \psi_0(\boldsymbol{x}) - \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,\chi_1^{(n-1)}(x', y')\,f(x', y') \tag{71a}$$

starting with $\psi^{(0)}(\boldsymbol{x}) = \psi_0(\boldsymbol{x})$. This formal calculation which illustrates once again the way of using the Fredholm integral equation approach, requires a more rigorous development, in particular $\partial_{z'}$ must be replaced by the normal derivative $\partial_{n'}$ to the rough plane, now in progress. In addition, this simple scattering problem has the virtue of making possible an interesting comparison with the conventional approach since then, the integral equation (11b) becomes on a rough plane

$$\psi(\boldsymbol{x}) = \psi_i(\boldsymbol{x}) - \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'\,[\psi_s(\boldsymbol{x}')\partial_{z'}g_D(\boldsymbol{x}, \boldsymbol{x}')]_{z'=f(x',y')}. \tag{72}$$

Then, assuming that g_D is the same as on a smooth plane, $\psi(\boldsymbol{x})$ may be obtained by guessing the form of $\psi_s(\boldsymbol{x}')$ [22, 23] on the plane $z' = f(x', y')$. For instance, in the case of periodic surfaces, the surface field in the integrand of (72) is expanded in terms of a Fourier series [24] which may be generalized to fractal surfaces [22], but this somewhat arbitrary choice also leads to tedious calculations and the correctness of the results is not guaranteed. One may also proceed as described in the remark of section 3.1.2.

The Fredholm integral formulation for scalar pulses, solutions of the wave equation and for electromagnetic waves will be discussed later.

Acknowledgment

I am indebted to Dr G Nurdin for numerical calculations.

Appendix A. Reflection of spherical waves on a mirror

We now consider a spherical wave $\psi_i(\boldsymbol{x})$ launched by a source at the point $(0, 0, -z_0)$ in the negative half-space $z < 0$ and impinging on a mirror in the plane $z = 0$,

$$\psi_i(\boldsymbol{x}) = |\boldsymbol{x}|^{-1} \exp(\mathrm{i}k|\boldsymbol{x}|) \qquad |\boldsymbol{x}|^2 = x^2 + y^2 + (z + z_0)^2. \tag{A1}$$

By symmetry, the reflected field $\psi_r(\boldsymbol{x})$ is a spherical wave originating from the image point $(0, 0, z_0)$ in the half-space $z > 0$. So

$$\psi_r(\boldsymbol{x}) = |\boldsymbol{y}|^{-1} \exp(\mathrm{i}k|\boldsymbol{y}|) \qquad |\boldsymbol{y}|^2 = x^2 + y^2 + (z - z_0)^2. \tag{A2}$$

We assume that the total field $\psi(\boldsymbol{x})$ satisfies the Neumann condition (6*b*), so we use the integral equation (7*b*) with $G_N(\boldsymbol{x}, \boldsymbol{x}')$ given by (8*a*) and (9). Then, we have from (23)

$$\left[4\pi^2 \partial_{z'} G_N(\boldsymbol{x}, \boldsymbol{x}')\right]_{z'=0} = -\iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma\, \exp[\mathrm{i}\beta(x - x') + \mathrm{i}\gamma(y - y')] \cos(k_z z) \tag{A3}$$

and according to the relations (A1) and (A2),

$$[\psi_{i,r}(\boldsymbol{x}')]_{z'=0} = \left(r^2 + z_0^2\right)^{-1/2} \exp\left[\mathrm{i}k\left(r^2 + z_0^2\right)^{1/2}\right] \qquad r^2 = x'^2 + y'^2. \tag{A4}$$

Taking into account (A3) and (A4), the integral equation (7*b*) may be written as

$$4\pi^2\left[\psi_i(\boldsymbol{x}) + \psi_r(\boldsymbol{x})\right] = 2A(\boldsymbol{x}) \tag{A5}$$

where $A(\boldsymbol{x})$ is defined in terms of the function $B(\beta, \gamma)$,

$$A(\boldsymbol{x}) = \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma\, \cos(k_z z) \exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\, B(\beta, \gamma) \tag{A6}$$

$$B(\beta, \gamma) = \iint_{-\infty}^{\infty} \mathrm{d}x'\, \mathrm{d}y'\, \left(r^2 + z_0^2\right)^{-1/2} \exp\left[-\mathrm{i}\beta x' - \mathrm{i}\gamma y' + \mathrm{i}k\left(r^2 + z_0^2\right)^{1/2}\right]. \tag{A7}$$

Introducing the polar coordinates $x' = r\cos\phi$, $y' = r\sin\phi$, the integral (A7) becomes

$$\begin{aligned} B(\beta, \gamma) &= \int_0^{\infty} r\, \mathrm{d}r\, \left(r^2 + z_0^2\right)^{-1/2} \exp\left[\mathrm{i}k\left(r^2 + z_0^2\right)^{1/2}\right] \int_0^{2\pi} \mathrm{d}\phi\, \exp[-\mathrm{i}r(\beta\cos\phi + \gamma\sin\phi)] \\ &= 2\pi \int_0^{\infty} r\, \mathrm{d}r\, \left(r^2 + z_0^2\right)^{-1/2} \exp\left[\mathrm{i}k\left(r^2 + z_0^2\right)^{1/2}\right] J_0\left[r\left(\beta^2 + \gamma^2\right)^{1/2}\right] \end{aligned} \tag{A8}$$

in which J_0 is the Bessel function of the first kind of order zero. $B(\beta, \gamma)$ is an integral of the Sonine–Gegenbauer type [12, 13] represented here by (A9),

$$\int_0^{\infty} t\, \mathrm{d}t\, J_0(bt)\left(t^2 - y^2\right)^{-1/2} \exp\left[\pm a\left(t^2 - y^2\right)^{1/2}\right] = \left(a^2 + b^2\right)^{-1/2} \exp\left[\pm \mathrm{i}y\left(a^2 + b^2\right)^{1/2}\right] \tag{A9}$$

the upper or lower sign is chosen accordingly as $a < 0$ or $a > 0$. But in (A8) $a = \mathrm{i}k$ is pure imaginary, so to apply (A9) we change k into $k + \mathrm{i}\varepsilon$, making ε tend to zero in the final result. So, we obtain

$$B(\beta, \gamma) = 2\mathrm{i}\pi\left(k^2 - \beta^2 - \gamma^2\right)^{-1/2} \exp\left[\mathrm{i}z_0\left(k^2 - \beta^2 - \gamma^2\right)^{1/2}\right]. \tag{A10}$$

Then, substituting (A10) into (A6) gives (since $k_z = \left(k^2 - \beta^2 - \gamma^2\right)^{1/2}$)

$$A(\boldsymbol{x}) = 2\mathrm{i}\pi \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma\, k_z^{-1} \cos(k_z z) \exp(\mathrm{i}\beta x + \mathrm{i}\gamma y + \mathrm{i}k_z z_0). \tag{A11}$$

Introducing the polar coordinates $\beta = t\cos\theta$, $\gamma = t\sin\theta$, the integral (A11) becomes

$$\begin{aligned}
A(\boldsymbol{x}) = 2\mathrm{i}\pi &\int_0^{\infty} t\,\mathrm{d}t\,\left(k^2 - t^2\right)^{-1/2} \exp\left[\mathrm{i}z_0\left(k^2 - t^2\right)^{1/2}\right] \cos\left[z\left(k^2 - t^2\right)^{1/2}\right] \\
&\times \int_0^{2\pi} \mathrm{d}\theta \exp[\mathrm{i}t(x\cos\theta + y\sin\theta)] \\
= 4\mathrm{i}\pi^2 &\int_0^{\infty} t\,\mathrm{d}t\,\left(k^2 - t^2\right)^{-1/2} \exp\left[\mathrm{i}z_0\left(k^2 - t^2\right)^{1/2}\right] \\
&\times \cos\left[z\left(k^2 - t^2\right)^{1/2}\right] J_0\left[t\left(x^2 + y^2\right)^{1/2}\right] \\
= 2\pi^2 &\int_0^{\infty} t\,\mathrm{d}t\,\left(t^2 - k^2\right)^{-1/2} f(t)\, J_0\left[t\left(x^2 + y^2\right)^{1/2}\right]
\end{aligned} \tag{A12}$$

in which the function $f(t)$ has the expression

$$f(t) = \exp\left[(z_0 + z)\left(t^2 - k^2\right)^{1/2}\right] + \exp\left[(z_0 - z)\left(t^2 - k^2\right)^{1/2}\right]. \tag{A13}$$

So, $A(\boldsymbol{x})$ is the sum of two Sonine–Gegenbauer integrals (A9) with, respectively, $a = z_0 + z$ and $a = z_0 - z$ with in both cases $a < 0$. Then, we obtain from (A9) and (A12) with $|\boldsymbol{x}|$, $|\boldsymbol{y}|$ defined by (A1) and (A2),

$$A(\boldsymbol{x}) = 2\pi^2\left[|\boldsymbol{x}| \exp(\mathrm{i}k|\boldsymbol{x}|) + |\boldsymbol{y}| \exp(\mathrm{i}k|\boldsymbol{y}|)\right] = 2\pi\left[\psi_i(\boldsymbol{x}) + \psi_r(\boldsymbol{x})\right] \tag{A14}$$

in agreement with (A5) which proves the consistency of the Fredholm equation.

Appendix B. Calculation of $\phi_i(\boldsymbol{u})$

According to the relations (36) and (37) in the main text

$$\begin{aligned}
4\pi\mathcal{G}(\boldsymbol{u}, \boldsymbol{u}') = \mathrm{i} &\int_{-\infty}^{\infty} \mathrm{d}\beta\, k_z^{-1} \exp[\mathrm{i}\beta(x - x')]\{\exp(\mathrm{i}k_z|z - z'|) + \exp(\mathrm{i}k_z|z + z'|)\} \\
&\times \exp(-N|z - z'|)
\end{aligned} \tag{B1}$$

and taking into account (23), a simple calculation gives

$$2\pi[\partial_{z'}\mathcal{G}(\boldsymbol{u}, \boldsymbol{u}')]_{z'=0} = -\int_{-\infty}^{\infty} \mathrm{d}\beta \exp[\mathrm{i}\beta(x - x')] \cos(k_z z)\left(1 + \mathrm{i}Nk_z^{-1}\right) \exp(Nz) \qquad z < 0 \tag{B2}$$

with according to (26)

$$[\psi_i(\boldsymbol{u}')]_{z'=0} = \exp(-\mathrm{i}kx'\sin\theta). \tag{B3}$$

Then, substituting (B2) and (B3) into (39a) and exchanging the integrals on x' and on β gives

$$\begin{aligned}
\phi_i(\boldsymbol{u}) = -(2\pi)^{-1} &\int_{-\infty}^{\infty} \mathrm{d}\beta \exp(\mathrm{i}\beta x) \cos(k_z z)\left(1 + \mathrm{i}Nk_z^{-1}\right) \exp(Nz) \\
&\times \int_{-\infty}^{\infty} \mathrm{d}x' \exp[-\mathrm{i}x'(\beta + k\sin\theta)] \\
= -&\int_{-\infty}^{\infty} \mathrm{d}\beta\, \delta(\beta + k\sin\theta) \exp(\mathrm{i}\beta x) \cos(k_z z)\left(1 + \mathrm{i}Nk_z^{-1}\right) \exp(Nz) \qquad z \leqslant 0
\end{aligned} \tag{B4}$$

in which δ is the Dirac distribution and finally since $k_z^2 = k^2 - \beta^2$, $N = d\varepsilon^{-1}(\varepsilon k^2 - \beta^2)$, we obtain

$$\phi_i(\boldsymbol{u}) = \exp(-\mathrm{i}kx\sin\theta)\cos(kz\cos\theta)\big[1 + \mathrm{i}kd(\varepsilon\sin\theta)^{-1}\big(\varepsilon - \sin^2\theta\big)\big] \times \exp\big[k^2 z d\varepsilon^{-1}\big(\varepsilon - \sin^2\theta\big)\big]. \tag{B5}$$

Appendix C. Calculation of the functions (61*a*) and (61*b*)

The functions (61*a*) and (61*b*) are defined by the integrals

$$A\{\psi_{i,r}\} = \iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\,\cos(k_z z)\exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\,H_{\beta,\gamma}\{\psi_{i,r}\} \tag{61a}$$

$$C\{\psi_{i,r}\} = \iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\,k_z^{-1}\cos(k_z z)\exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\,H_{\beta,\gamma}\{\psi_{i,r}\}. \tag{61b}$$

When ψ_i is the spherical wave (A1) of appendix A, we have $H_{\beta,\gamma}\psi_i\} = B(\beta,\gamma)$, where $B(\beta,\gamma)$ is the function (A7) so that substituting (A9) into (61*a*) gives the expression (A14) which we write as

$$A\{\psi_i\} = 2\pi^2\psi^{(0)}(\boldsymbol{x}) \qquad \psi^{(0)}(\boldsymbol{x}) = \psi_i(\boldsymbol{x}) + \psi_r^{(0)}(\boldsymbol{x}) \tag{C1}$$

with $\psi_r^{(0)}$ denoting the reflected field (A2) at a perfectly reflecting smooth plane. Substituting (A10) into (61*b*) and still using the polar coordinates $\beta = t\cos\theta$, $\gamma = t\sin\theta$ we obtain

$$C\{\psi_i\} = 2\pi^2\int_0^{\infty} t\,\mathrm{d}t\,\big(t^2 - k^2\big)^{-1} f(t)\,J_0\big[t\big(x^2 + y^2\big)^{1/2}\big] \tag{C2}$$

which is the integral (A12) except that $\big(t^2 - k^2\big)^{-1/2}$ in front of $f(t)$ is changed into $\big(t^2 - k^2\big)^{-1}$. But, taking into account the definition (A13) of $f(t)$, one easily obtains from the derivative $\partial_{z^\circ} f(t)$ the relation (for typographical reasons we write z° for z_0) $A\{\psi_i\} = \partial_{z^\circ}C\{\psi_i\}$. Then, according to the relation (C1) and introducing the antiderivative (primitive) symbol $\partial_{z^\circ}^{-1}$ we obtain

$$C\{\psi_i\} = 2\pi^2\partial_{z^\circ}^{-1}\psi^{(0)}(\boldsymbol{x}). \tag{C3}$$

Finally, substituting (C1) and (C2) into (60*a*) gives

$$2\phi\{\psi_i\} = \alpha\big(1 + aZ\partial_{z^\circ}^{-1}\big)\psi^{(0)}(\boldsymbol{x}) \qquad \alpha = \exp(-\mathrm{i}aZz) \tag{C4}$$

which determines the right-hand side of (60).

Appendix D. First iterated term of the Rayleigh–Gans process

Starting with $\chi^{(0)}(\boldsymbol{x}) = \Phi\big\{\psi^{(0)}\big\}$ we obtain for the first step of the iterative process (68)

$$\chi^{(1)}(\boldsymbol{x}) = \phi\{\Phi\} + \Phi\big\{\psi^{(0)}\big\} \tag{D1}$$

with according to (67*a*) (since from the relations (60)–(62) $\phi(\psi_1 + \psi_2) = \phi(\psi_1) + f(\psi_2)$),

$$\phi\{\Phi\} = (\alpha - 1)\phi\big\{\psi^{(0)}\big\} + \alpha aZ\phi\big\{\partial_{z^\circ}^{-1}\psi^{(0)}\big\}. \tag{D2}$$

Then, we obtain from (64*a*) and from the relation (C4) of appendix C,

$$\phi\{\psi^{(0)}\} = 2\phi\{\psi_i\} = \alpha\big(1 + \alpha aZ\partial_{z^\circ}^{-1}\big)\psi^{(0)} \tag{D3}$$

while to obtain the second term in (D3) we first need to calculate $H_{\beta,\gamma}\{\partial_{z^\circ}^{-1}\psi^{(0)}\}$. Still using the relation $[\psi^{(0)}]_{z'=0} = 2[\psi_i]_{z'=0}$ and exchanging the integrations on x', y' and on z_0, we obtain from (62) and from the relation (A10) of appendix A

$$H_{\beta,\gamma}\left\{\partial_{z^\circ}^{-1}\psi^{(0)}\right\} = 2\partial_{z^\circ}^{-1}H_{\beta,\gamma}\{\psi_i\} = 2\partial_{z^\circ}^{-1}B(\beta,\gamma) = -2\mathrm{i}k_z^{-1}B(\beta,\gamma). \tag{D4}$$

Substituting (D4) into (61a) and (61b) gives

$$\begin{aligned} A\left\{\partial_{z^\circ}^{-1}\psi^{(0)}\right\} &= -2\mathrm{i}\iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\, k_z^{-2}\cos(k_z z)\exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\,B(\beta,\gamma) \\ C\left\{\partial_{z^\circ}^{-1}\psi^{(0)}\right\} &= -2\mathrm{i}\iint_{-\infty}^{\infty} \mathrm{d}\beta\,\mathrm{d}\gamma\, k_z^{-3}\cos(k_z z)\exp(\mathrm{i}\beta x + \mathrm{i}\gamma y)\,B(\beta,\gamma). \end{aligned} \tag{D5}$$

Using the expression (A10) of $B(\beta,\gamma)$ and the polar coordinates $\beta = t\cos\theta$, $\gamma = t\sin\theta$ the integrals (D5) take the form (A12) with the factor $\left(t^2 - k^2\right)^{-1/2}$ in front of $f(t)$ changed, respectively, into $\left(t^2 - k^2\right)^{-1}$ and $\left(t^2 - k^2\right)^{-3/2}$. Then, taking the first and second derivatives with respect to z_0 of $f(t)$, we obtain according to (A13) and (C1),

$$A\left\{\partial_{z^\circ}^{-1}\psi^{(0)}\right\} = -2\mathrm{i}\partial_{z^\circ}^{-1}A\{\psi_i\} = -4\mathrm{i}\pi^2\partial_{z^\circ}^{-1}\psi^{(0)} \tag{D6}$$

$$C\left\{\partial_{z^\circ}^{-1}\psi^{(0)}\right\} = -2\mathrm{i}\partial_{z^\circ}^{-2}A\{\psi_i\} = -4\mathrm{i}\pi^2\partial_{z^\circ}^{-2}\psi^{(0)}. \tag{D7}$$

Substituting (D6) and (D7) into (60a) gives

$$\phi\left\{\partial_{z^\circ}^{-1}\psi^{(0)}\right\} = -\mathrm{i}\alpha\left(1 + \mathrm{i}\alpha a Z\partial_{z^\circ}^{-1}\right)\partial_{z^\circ}^{-1}\psi^{(0)} \tag{D8}$$

and from (D2), (D3) and (D8) we obtain

$$\phi\{\chi^{(0)}\} = \alpha(\alpha-1)\left(1 + \alpha a Z\partial_{z^\circ}^{-1}\right)\psi^{(0)} - \mathrm{i}\alpha^2 a Z\left(1 + \alpha a Z\partial_{z^\circ}^{-1}\right)\partial_{z^\circ}^{-1}\psi^{(0)}. \tag{D9}$$

So, according to (D1) and (D9), the first iterated term for the impedance contribution to the solution of equation (60) is with $\Phi\left\{\psi^{(0)}\right\}$ given by (67a),

$$\chi^{(1)}(\boldsymbol{x}) = \Phi\left\{\psi^{(0)}\right\} + \alpha(\alpha-1)\left(1 + \alpha^2 a Z\partial_{z^\circ}^{-1}\right)\psi^{(0)}(x) - \mathrm{i}\alpha a Z\left(1 + \alpha a Z\partial_{z^\circ}^{-1}\right)\partial_{z^\circ}^{-1}\psi^{(0)}(\boldsymbol{x}) \tag{D10}$$

an expression that depends on $aZ\partial_{z^\circ}^{-1}\psi^{(0)}$ and $a^2Z^2\partial_{z^\circ}^{-2}\psi^{(0)}$.

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